Final Exam
Name:_____

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **Only scientific calculators are allowed on this exam.**

Show your work!

1. (8 points) Find the critical points and the draw the phase diagram for the autonomous differential equation

$$\frac{dx}{dt} = (x^2 - 5x + 4)(x^2 - 16)$$

and identify which critical points are stable and unstable.

2. (10 points) Consider a population of giraffes which initially has 100 giraffes at time $t = 0$. Assume also that the population is governed by the logistic equation

$$\frac{dP}{dt} = (0.005)P(1000 - P).$$

Find the population P as a function of t (in years) and determine how long it will take the giraffe population to reach 90% of its limiting population.

3. (10 points) Find the general solution to the differential equation

$$(\cos x + \ln y)dx + \left(\frac{x}{y} + e^y\right) dy = 0.$$

4. (10 points) A tank initially contains 120 gallons of pure water. Brine containing 2 lb of salt per gallon enters the tank at a rate of 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min. Determine when the tank is empty and find the amount of salt in the tank after t minutes.

5. (10 points) Consider a body that moves horizontally through a medium so that its acceleration is proportional to the square of velocity so that

$$\frac{dv}{dt} = -20v^2.$$

Assuming that $v(0) = 1$ and $x(0) = 10$, find the position $x(t)$ as a function of t.

6. (10 points) Find the general form of the complementary solution of the differential equation

$$6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4 = 0$$

which has characteristic function

$$(r^2 + 4)(6r^2 + 5r + 1) = 0.$$

7. (12 points) A bridge (not far outside Manchester) has a platoon of soldiers walking across it in cadence. Assume that there is no damping and that the natural frequency of the bridge is $\omega_0 = 10$. Assume also that the frequency of the soldiers cadence is such that the external force is given by $F(t) = 225\cos 5t + 300\sin 5t$ so that the associated differential equation is given by

$$x'' + 100x = 225\cos 5t + 300\sin 5t$$

with initial conditions $x(0) = 375$ and $x'(0) = 0$. Find the unique solution which governs the motion of the bridge. For a bonus two points, explain why the soldiers are or are not in danger of the bridge collapsing.

(Extra work space for 7.)

8. (10 points) Find the general solution to the system

$$x' = 3x - y, \quad y' = 5x - 3y.$$

9. (10 points) Use the fact that $\mathcal{L}\{tf(t)\} = -F'(s)$ to find the Laplace transform of

$$tx'' - 2x' + tx = 0; \quad x(0) = 0.$$

For a bonus 5 points, solve the transform for $X(s)$.

10. (10 points) Use the fact that $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ to find the Laplace transform of

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ t^3 & \text{if } t \geq 2. \end{cases}$$

Bonus Question. (10 points) Solve the differential equation

$$x'' + 2x' + 26x = 0; \quad x(0) = 3, x'(0) = 10.$$